

TRANSPORT FINDINGS

When Are Impedance Choices Irrelevant? Equivalence Conditions for Hansen-Style Access Metrics

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Findings

Access indices often differ only by the impedance function f that downweights opportunities by travel cost. We give conditions under which different impedances (i) yield the same percentage responses in a standard planar benchmark, (ii) give levels that differ only by a multiplicative constant, and (iii) preserve cross-place rankings. We work fully in discrete form, so results apply to finite opportunity sets without measure notation.

1. Questions

Hansen (1959) established the weighted-sum structure of generalized cost components (e.g., time, money, operating costs) that underlies many access indices. In practice, analysts must pick an impedance f (e.g., a cumulative threshold (all opportunities within T equally counted), an exponential decay ($e^{-\theta t}$), or an inverse power ($1/t^p$)).

We ask:

1. When do different impedances yield the same percentage changes in access under uniform changes in land use, speed, or time budget?
2. When are cross-place rankings independent of the impedance?
3. How do these conditions relate to the frequent empirical finding that cumulative and gravity-type metrics are highly correlated?

Empirically, many studies report that common access measures often yield similar rankings or conclusions under typical parameter choices (Santana Palacios and El-Geneidy 2022; Kapatsila et al. 2023; Vale and Pereira 2017). This motivates the hypothesis that, under mild conditions, impedance choice may be irrelevant in specific senses. However, the precise reasons for this robustness have not been formally established. This paper sharpens and extends earlier insights by deriving general analytic results. We provide explicit elasticity formulas for accessibility with respect to uniform changes in land-use density, travel speed, and time budget, showing how these sensitivities compare across impedance functions (building on the qualitative trade-offs

noted by Levine et al. (2012)). We also introduce a new decomposition of Hansen’s index into discrete distance-band contributions, which offers a clearer interpretation of how different decay functions weight near vs. far opportunities. Finally, we identify a simple dominance condition – proportional cumulative opportunity curves across locations – under which any impedance choice will yield identical cross-place accessibility rankings. This theoretical condition helps explain the empirical findings of prior work that different accessibility metrics often lead to the same conclusions. In these ways, our paper formalises and generalises scattered observations from the literature, providing a stronger foundation for when impedance choices are truly irrelevant and when they might matter.

2. Methods

Note: A complete nomenclature table is provided in [Table 1](#).

We work with travel cost or time, denoted t or T for arguments and budgets, respectively. Here, ‘budget’ means a time or generalised cost limit or threshold, not monetary expenditure. Let c_{ij} be generalised cost from origin i to opportunity j , and $O_j \geq 0$ the size of opportunity j .

Hansen’s primal accessibility with budget T is given by

$$A_i(T; f) = \sum_{j: c_{ij} \leq T} O_j f(c_{ij}). \quad (1)$$

We use this budgeted form for all impedances. While an unbudgeted sum exists when $\int_0^\infty t f(t) dt < \infty$ in the uniform-plane benchmark, which assumes constant speed across space (no congestion or signals), discrete sums are typically budgeted in practice, using the cumulative curve $N_i(t)$ defined as the sum of opportunities that can be reached from i within travel cost t :

$$N_i(t) = \sum_{j: c_{ij} \leq t} O_j. \quad (2)$$

Discrete jump-sum identity. The jump-sum identity says that accessibility can be computed as the sum, over each travel-time threshold, of the cumulative number of reachable opportunities at that threshold multiplied by the drop in the impedance weight at that point. Formally: Assume f is non-increasing, bounded, and right-continuous with $f(\infty) = 0$, so $\Delta f_k \geq 0$ is well defined. Let $t_1 < \dots < t_m$ be the strictly increasing ordering of the distinct costs $\{c_{ij}\}$ where $N_i(t)$ jumps, with jumps $J_i(t_k) = \sum_{j: c_{ij} = t_k} O_j$, and set $t_{m+1} = \infty$. Define $\Delta f_k := f(t_k) - f(t_{k+1})$ with $f(\infty) = 0$. Then

$$A_i(T; f) = \sum_{k: t_k \leq T} N_i(t_k) \Delta f_k. \quad (3)$$

Note: Any tie-breaking that preserves the cumulative steps yields the same A_i , since only $N_i(t_k)$ and Δf_k enter Equation (3). The jump-sum identity is shown in Appendix A.

Because f is non-increasing, the framework excludes cases where being closer is undesirable (e.g., noise/safety next to certain stops).

Uniform-plane benchmark. The uniform-plane benchmark assumes opportunities are spread evenly across space and travel occurs at constant speed, so accessibility reduces to integrating opportunity density over concentric circles around an origin. Formally: for a uniform planar density ρ and speed v , the marginal annulus area at time t is proportional to t , so

$$A_i(T; f) = 2\pi\rho v^2 \int_0^T t f(t) dt. \quad (4)$$

This gives the elasticities

$$E_\rho = 1, \quad E_v = 2, \quad E_T(T; f) = \frac{T^2 f(T)}{\int_0^T t f(t) dt}. \quad (5)$$

Ranking invariance across impedances. If the cumulative opportunity curves for two places are proportional at all travel times, then their accessibility rankings will be the same no matter which impedance function is used. More precisely: if cumulative curves are proportional across places, $N_i(t) = \alpha_i N^*(t)$ for all t , then for any non-increasing f with $f(\infty) = 0$,

$$A_i(T; f) = \alpha_i \sum_{k:t_k \leq T} N^*(t_k) \Delta f_k, \quad (6)$$

so $A_i > A_{i'}$ if and only if $\alpha_i > \alpha_{i'}$, independent of f .

3. Findings

We next discuss the Findings in order.

Three results explain when impedance choice matters:

- First, in a uniform planar benchmark, density and speed elasticities are invariant across impedances.
- Second, cumulative opportunities is the only such impedance function with a constant time elasticity. See Eq. (7).
- Third, if cumulative curves are proportional across places, cross-place rankings do not depend on the impedance.

Table 1. Nomenclature.

Symbol	Units	Meaning
$A_i(T; f)$	opportunities	Hansen access at i under impedance f .
c_{ij}	minutes or cost	Generalised travel cost from origin i to opportunity j .
E_T, E_v, E_ρ	dimensionless	Elasticities with respect to T , v , and ρ .
$f(c)$	dimensionless	Impedance function, non-increasing in c .
Δf_k	dimensionless	Weight increment $f(t_k) - f(t_{k+1})$ with $f(\infty) = 0$.
k	index	Threshold index for t_k (distinct sorted costs where $N_i(t)$ jumps).
O_j	opportunities	Size of opportunity at j (e.g., jobs).
$N_i(t)$	opportunities	Cumulative number of opportunities reachable from i within t .
$N^*(t)$	opportunities	Reference cumulative curve used in proportionality statements.
T	minutes or cost	Travel time or cost budget.
t	minutes or cost	Travel time or cost argument.
t_0	minutes or cost	Micro-cutoff parameter in the inverse-square impedance.
t_k	minutes or cost	Distinct thresholds where $N_i(t)$ jumps.
t^*	minutes or cost	Typical travel time used for local comparisons.
v	distance/time	Travel speed in the uniform-plane benchmark.
α_i	dimensionless	Proportionality constant with $N_i(t) = \alpha_i N^*(t)$.
θ	per minute or per dollar	Decay parameter in $f(t) = e^{-\theta t}$.
ρ	opp./area	Opportunity density in the uniform-plane benchmark.

F1. Elasticity equivalence for land use and speed. On the uniform plane (see Methods, Section 2), the marginal annulus area at time t is proportional to t , implying $E_\rho = 1$, $E_v = 2$ (see Eq. (5)).

These invariances rely on the uniform-plane benchmark; strong spatial non-uniformity or network effects can break them.

F2. Constant time elasticity. Among the canonical impedances shown here, only the cumulative opportunities has $E_T \equiv \text{constant}$. For cumulative opportunities $f(t) = \mathbf{1}\{t \leq T\}$,

$$A = \pi \rho v^2 T^2, \quad E_T = 2 \quad \forall T. \quad (7)$$

Other impedances have $E_T(T; f)$ that varies with T , determined by the tail of f .

F3. Ranking invariance across impedances. If cumulative curves are proportional across places, $N_i(t) = \alpha_i N^*(t)$ for all t , then for any non-increasing f with $f(\infty) = 0$,

$$A_i(T; f) = \alpha_i \sum_{k: t_k \leq T} N^*(t_k) \Delta f_k, \quad (8)$$

so $A_i > A_{i'}$ if and only if $\alpha_i > \alpha_{i'}$, independent of f .

To diagnose whether this holds: plot $N_i(t)/N_i(T)$ for competing places over $t \in [0, T]$; nearly parallel curves indicate ranking stability across impedances, while crossings signal possible ranking flips as θ or T varies.

F4. Practical indifference in narrow bands. When relevant travel times concentrate near a typical t^* , many impedances are nearly flat on that band, so percentage responses and rankings are often locally similar.

Illustration on the uniform plane. Three canonical impedances give closed forms with compact displays. Cumulative $f(t) = \mathbf{1}\{t \leq T\}$:

$$A = \pi \rho v^2 T^2, \quad E_T = 2. \quad (9)$$

Exponential $f(t) = e^{-\theta t}$, where θ denotes the exponential decay parameter:

$$A = \frac{2\pi \rho v^2}{\theta^2} \left[1 - (1 + \theta T)e^{-\theta T} \right], \quad (10)$$

$$E_T(T; e^{-\theta \cdot}) = \frac{\theta^2 T^2 e^{-\theta T}}{1 - (1 + \theta T)e^{-\theta T}}.$$

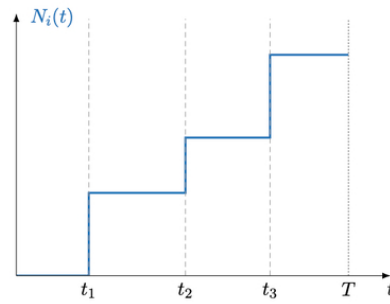
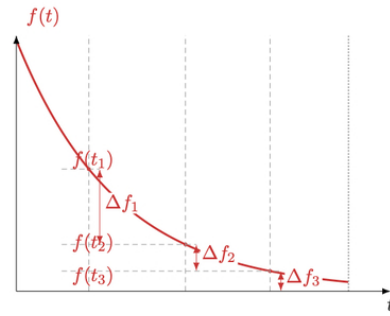
The micro-cutoff t_0 is the smallest practical time considered, used to avoid singularities at $t = 0$. Inverse-square with micro-cutoff $f(t) = \mathbf{1}\{t \geq t_0\} (t_0/t)^2$:

$$A = 2\pi \rho v^2 t_0^2 \ln \frac{T}{t_0}, \quad E_T(T; f) = \frac{1}{\ln(T/t_0)}. \quad (11)$$

In all three cases $E_\rho = 1$ and $E_v = 2$ (see Eq. (5)).

When equivalence fails. See Supplemental Information for worked examples. *Ranking invariance* requires $N_i(t) = \alpha_i N^*(t)$ for all t . If curves cross, weights on near versus far thresholds matter and different f can reverse rankings, and thus interpretations. *Time-elasticity* invariance relies on the planar annulus area being proportional to t , so non-uniform density or network effects can break it.

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(a) Cumulative opportunities $N_i(t)$.(b) Impedance $f(t)$, with baselines $f(t_k)$ and drops Δf

$$\sum_k N_i(t_k) \Delta f_k = A_i(T; f)$$

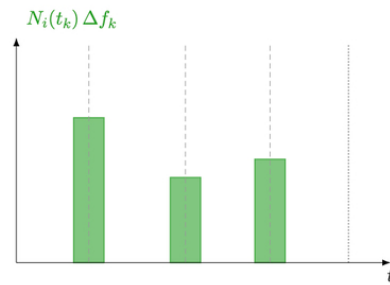
(c) Contributions $N_i(t_k) \Delta f_k$ that sum to $A_i(T; f)$.

Figure 1. Illustration of the discrete jump–sum identity (Equation (3)) with three cost thresholds $t_1 < t_2 < t_3 \leq T$. Top panel: cumulative opportunities $N_i(t)$. Middle panel: impedance $f(t)$ with baselines $f(t_k)$. Bottom panel: contributions $N_i(t_k) \Delta f_k$. Red segments $f(t_1)$, $f(t_2)$, and $f(t_3)$ determine the baselines of Δf_k . Green bars $N_i(t_k) \Delta f_k$ at bottom add to $A_i(T; f)$ under the budget T .



REFERENCES

- Hansen, Walter G. 1959. "How Accessibility Shapes Land Use." *Journal of the American Institute of Planners* 25 (2): 73–76. <https://doi.org/10.1080/01944365908978307>.
- Kapatsila, Bogdan, Manuel Santana Palacios, Emily Gris , and Ahmed El-Geneidy. 2023. "Resolving the Accessibility Dilemma: Comparing Cumulative and Gravity-Based Measures of Accessibility in Eight Canadian Cities." *Journal of Transport Geography* 107:103530. <https://doi.org/10.1016/j.jtrangeo.2023.103530>.
- Levine, Jonathan, Joe Grengs, Qingyun Shen, and Qing Shen. 2012. "Does Accessibility Require Density or Speed?" *Journal of the American Planning Association* 78 (2): 157–72. <https://doi.org/10.1080/01944363.2012.677119>.
- Santana Palacios, Manuel, and Ahmed El-Geneidy. 2022. "Cumulative versus Gravity-Based Accessibility Measures: Which One to Use?" *Findings*. <https://doi.org/10.32866/001c.32444>.
- Vale, David S., and M rio Pereira. 2017. "The Influence of the Impedance Function on Gravity-Based Pedestrian Accessibility Measures: A Comparative Analysis." *Environment and Planning B: Urban Analytics and City Science* 44 (4): 740–63. <https://doi.org/10.1177/0265813516641685>.

A. Supplemental Information: Examples

A.1. Ranking flip threshold (what makes rankings change?)

Motivation. The main text shows when rankings are invariant. This example isolates the opposite case, when two places can swap order as the impedance shifts attention from near to far opportunities.

Place A has $N_{A,\text{near}}$ opportunities at cost a and $N_{A,\text{far}}$ at cost $b > a$. Place B has $N_{B,\text{near}}$ at a and $N_{B,\text{far}}$ at b . With the exponential impedance $f(t) = e^{-\theta t}$,

$$A_A(\theta) - A_B(\theta) = (N_{A,\text{near}} - N_{B,\text{near}})e^{-\theta a} + (N_{A,\text{far}} - N_{B,\text{far}})e^{-\theta b}. \quad (12)$$

If the near advantage favors A and the far advantage favors B,

$$N_{A,\text{near}} > N_{B,\text{near}}, \quad N_{A,\text{far}} < N_{B,\text{far}}, \quad (13)$$

to ensure $\theta^* > 0$, require $(N_{B,\text{far}} - N_{A,\text{far}}) > (N_{A,\text{near}} - N_{B,\text{near}})$; then there is a unique threshold θ^* where the ranking flips,

$$\theta^* = \frac{1}{b-a} \ln \left(\frac{N_{B,\text{far}} - N_{A,\text{far}}}{N_{A,\text{near}} - N_{B,\text{near}}} \right). \quad (14)$$

For $\theta > \theta^*$ the near opportunities dominate and $A_A(\theta) > A_B(\theta)$, for $\theta < \theta^*$ the far opportunities dominate and $A_B(\theta) > A_A(\theta)$.

For the cumulative impedance with budget T , the ranking is piecewise by threshold: if $T < a$ both are zero, if $a \leq T < b$ the near counts decide, if $T \geq b$ the near plus far counts decide.

A.2. When different impedances give almost the same answer

Motivation. Practitioners often care about a specific time range, for example typical peak travel times. Many common impedances place most of their weight on that range. If two places look similar within that range, their access scores will be nearly proportional, no matter which such impedance is used.

Let the times where another opportunity becomes reachable be $\{t_k\}$ up to budget T , and write the discrete representation

$$\begin{aligned} A_i(T; f) &= \sum_{k:t_k \leq T} N_i(t_k) \Delta f_k, \\ \Delta f_k &= f(t_k) - f(t_{k+1}), \\ f(\infty) &= 0. \end{aligned} \quad (15)$$

Fix a time range of interest $[a, b] \subset [0, T]$ and split the sum into inside and outside that range,

$$A_i(T; f) = \underbrace{\sum_{t_k \in [a, b]} N_i(t_k) \Delta f_k}_{\text{in range}} + \underbrace{\sum_{t_k \notin [a, b]} N_i(t_k) \Delta f_k}_{\text{outside range}}. \quad (16)$$

When an impedance concentrates on a time range where the places look alike, different impedances make little difference to the comparison. More formally:

If most of the impedance weight lies in $[a, b]$,

$$\sum_{t_k \notin [a, b]} \Delta f_k \leq \varepsilon \quad \text{with small } \varepsilon, \quad (17)$$

and the two places have approximately proportional cumulative curves on that same range,

$$N_A(t) \approx \alpha N_B(t) \quad \forall t \in [a, b], \quad (18)$$

then their scores are nearly proportional,

$$A_A(T; f) \approx \alpha A_B(T; f). \quad (19)$$

This leaves a discrepancy no larger than ε times a bound on the cumulative counts.

SUPPLEMENTARY MATERIALS

SI

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