

TRANSPORT FINDINGS

Principles of Least Action in Urban Traffic Flow

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Transport Findings

Drawing on the strong connection between traffic modelling and physics we apply the principle of least action to traffic flow in an urban setting.

RESEARCH QUESTIONS AND HYPOTHESIS

In this article we draw on the strong connection between traffic modelling and physics to discuss the following two questions:

- How can the principle of least action be used to consider traffic flow in an urban setting?
- How can this formulation be expanded to include stochastic effects analogous to quantum mechanical effects to capture phenomenon not typically found in traffic models?

METHODS AND DATA

In Fermat's principle of least time, vehicles move between two points in space (A and B) in the least amount of time possible.¹ We consider a total travel time as a generalized principle of least action:

$$\mathcal{T}[z] = \int_A^B \tau(z) dz \quad (1)$$

for an instantaneous travel time $\tau(z)$, and a total travel time $\mathcal{T}[z]$. This travel time can be generalized to a cost function if desired, so long as the units remain consistent throughout the problem. The problem can also consider vehicle speed along a path z , by understanding that an instantaneous path length dz , divided by the speed $u(z)$ at any point produces the instantaneous travel time, or

$$\mathcal{T}[z] = \int_A^B \frac{dz}{u(z)} \quad (2)$$

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¹ An interested reader can consult the *Feynman Lectures on Physics* (Feynman, Leighton, and Sands 2011a, 2011b) for more detail.

The speed function $u(z)$ can be related to a volume delay function in traffic assignment modelling. In a direct analogy to physics, a volume delay function can be thought of as an application of a potential that affects route choice and the willingness to make a trip.

Under the principle of stationary action, a set of equations known as the Euler-Lagrange equations can be derived for each of the generalized coordinates z_a . These equations have the form

$$t\tau(z)\dot{z} - \tau(z)z = 0 \quad (3)$$

Application of the Euler-Lagrange equation presented in equation 3 can be used to find the path $z(t)$ that minimizes $\mathcal{T}[z]$ in equation 2. This formulation is applied to an idealized situation of traffic flow in a city experiencing radially symmetric congestion. The intention is not to accurately model common scenarios, but as a proof of concept outlining the process required to retain results for the small set of situations that produce analytical functions. More complex situations can be evaluated numerically if desired.

After this classical application is outlined, we provide qualitative discussion of how the classical approach can be extended to quantum mechanical applications in traffic modelling.

FINDINGS

Consider a situation in which peak period travel has caused radially symmetric congestion in a city. Congestion is highest at the origin (downtown), and decreases radially outwards. A driver wishes to move between two arbitrary points in the least amount of time possible, given the congestion described above. For this problem we apply Fermat's principle.

Due to the radially symmetric nature of the problem, the polar coordinate system is chosen as the most suitable. The coordinates (r, ϕ) are given in terms of standard Cartesian coordinates as $x = r \cos(\phi)$ and $y = r \sin(\phi)$. The standard unit of infinitesimal distance is given as

$$da = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2 d\phi^2} = \sqrt{1 + r^2 \phi'^2} dr \quad (4)$$

where $\phi' = d\phi/dr$. Given that travel is made between two points in the shortest time, equation 4 is applied to equation 2 to produce

$$\mathcal{T} = \int_A^B dr \frac{\sqrt{1 + r^2 \phi'^2}}{u(r, \phi)} \quad (5)$$

At this point a function for the speed is necessary. For analytical simplicity, suppose that the congestion affects travel speed as

$$u(r) = \alpha \sqrt{r} \quad (6)$$

for some arbitrary constant α . This is in fact the classic Brachistochrone problem in polar coordinates, and a radial gravitational field. Using equation 6, equation 5 becomes

$$\mathcal{T} = \int_A^B dr \frac{\sqrt{1 + r^2 \phi^2}}{\alpha \sqrt{r}}$$

The Euler-Lagrange equation in this case is

$$rG\phi' - G\phi = 0$$

where

$$G = \frac{\sqrt{1 + r^2 \phi^2}}{\alpha \sqrt{r}}$$

The resulting differential equation is

$$r \left[\frac{r^{3/2} \phi'}{\alpha \sqrt{1 + r^2 \phi'^2}} \right] = 0$$

which implies that

$$\frac{r^{3/2} \phi'}{\alpha \sqrt{1 + r^2 \phi'^2}} = \frac{a}{\alpha}$$

For some arbitrary constant a . This is itself a differential equation of the form

$$\phi'(r) = \frac{a}{\sqrt{r^3 - a^2 r^2}}$$

with a solution

$$\phi(r) = 2 \arctan\left(\frac{\sqrt{r - a^2}}{a}\right) + C$$

or, alternatively,

$$r(\phi) = a^2 \sec^2\left(\frac{\phi}{2} - C\right)$$

The constant a can be partially determined by requiring the function to pass through initial coordinates (r_1, ϕ_1) . The first point can be used to find

$$a = \sqrt{r_1} \cos\left(\frac{\phi_1}{2} - C\right)$$

leaving only the phase angle C to be determined, or

$$r(\phi) = r_1 \cos^2\left(\frac{\phi_1}{2} - C\right) \sec^2\left(\frac{\phi}{2} - C\right)$$

C is varied numerically and the trajectory is examined to determine if it passes through the desired destination point. Varying C rotates the trajectory function about (r_1, ϕ_1) . To restrict trajectories to fall in only one hemicircle, and eliminate duplicate solutions, C is restricted to

$$-\pi \leq C < \pi$$

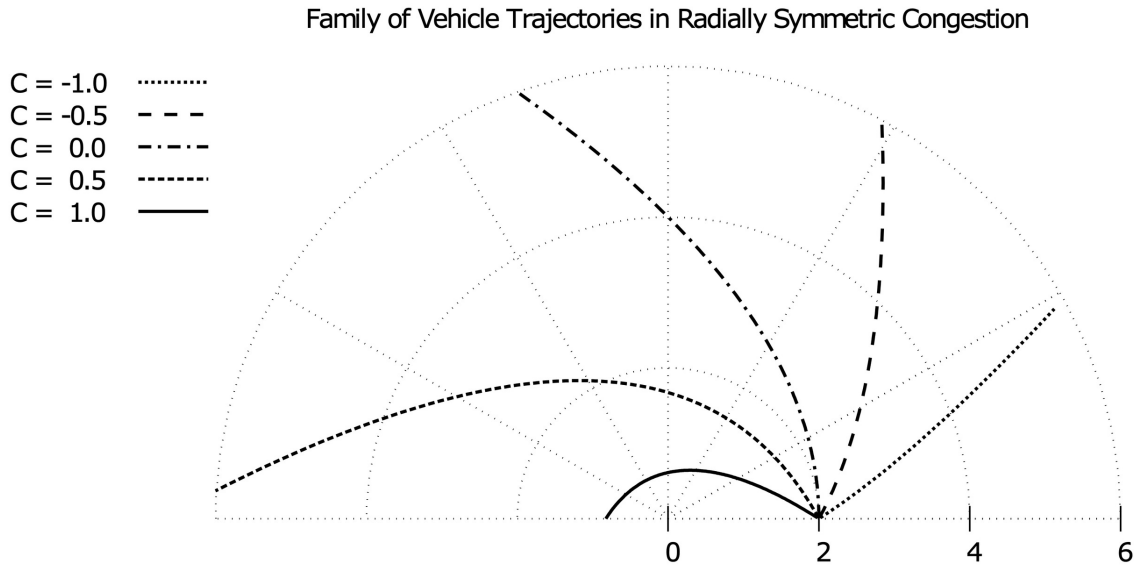


Figure 1: Sample vehicle trajectories with the same origin point.

One mathematical artifact of the simplified model is that trajectories cannot pass through the origin, and in fact trajectories must always pass *around* the origin (even for trips made along the same ray towards or away from the origin). This is a result of the singularity that appears due to the simplicity of the model, and may not appear if another, less analytical model of congestion is used. The important conclusion is that faced with this congestion, drivers will not follow a straight line between their origin and destination. Instead, drivers “skirt” the core and make their way around it in some way. Figure 1 provides some example trajectories for a specified origin of $(2, 0)$, for the upper hemimcircle, with the understanding that the lower hemimcircle is identical when reflected about the lines $\phi = 0$ and $\phi = \pi$ (the horizontal axis).

TRIP ENERGY THRESHOLD

Einstein (1905) demonstrated that the energy required to knock electrons from a metal is discretized; a certain energy threshold is required to begin removing the electrons. Similarly, there is a certain energy threshold that is required for trip to be worthwhile; below that energy no trips will happen. This energy threshold is the amount of effort required by a driver to make *any* trip, regardless of the length or obstacles faced. This parallels the concept of utility in travel, and makes the argument that travel behaviour changes in discrete, quantized steps. A trip energy threshold could be quantified for a given region or city using a logit or distribution typical in the literature for modelling trip choice. In a recent thesis, Hancock (2019) proposed a quantum-state approach to modelling choices, indicating that incorporating quantum ideas into transportation modelling is of continued interest to scholars.

DEVIATION FROM A SINGLE PATH

In quantum mechanical formulations, the path derived from the principle of stationary action as above is allowed to deviate at a cost, which can accommodate trips that do not strictly follow the path of least time. This accounts for drivers who do not take the optimal route, whether out of habit or inertia (Xu et al. 2017; Srinivasan and Mahmassani 2000). Each individual path takes the form of a probability distribution with decreasing likelihood based on increased distance from that of least action. This creates a distribution of possible paths that can be combined to form a distribution of possible states and paths used by individuals, capturing the random aspects of human behaviour that do not always follow a mathematically optimal path. This type of approach was used by Feynman (1942) to bridge classical and quantum mechanics; this approach could also be applied to Lagrangian models such as the one presented here to develop a quantum-analogous state space for travel and congestion in a city.



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